Lecture 21 on Dec. 02 2013

This is the last note of this course and we are going to see how can we apply the residue theorem introduced in lecture 20. The main application is to evaluating the definite integral of one variable real functions. There are four types of integrals that we are going to study. We introduce them one by one in the following arguments.

Type I. Letting $R(x)$ be a rational function (same assumption is used in Type II, III and IV), we evaluate

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta. \tag{0.1}$$

Strategy: Assume $z = e^{i\theta}$. Then while θ runs from 0 to 2π, z runs along the unit circle counterclockwisely. by this change of variable, we know that

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \qquad \mathrm{d}z = e^{i\theta}\,i\,\mathrm{d}\theta = i\,z\,\mathrm{d}\theta.$$

With the above equalities, (0.1) can be rewritten as

$$\int_{|z|=1} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{\mathrm{d}z}{iz}.$$

Therefore the residue theorem can be applied.

Example 1. Evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta},$$

where a is a real number satisfying $|a| > 1$.

Solution: By the strategy, we know that the integral can be transformed to

$$\int_{|z|=1} \frac{1}{a + \frac{1}{2}(z+\frac{1}{z})}\frac{\mathrm{d}z}{iz} = \int_{|z|=1}\frac{-2i}{z^2+2az+1}\,\mathrm{d}z.$$

$z^2 + 2az + 1$ has two roots. They are

$$z_1 = -a + \sqrt{a^2-1}, \qquad z_2 = -a - \sqrt{a^2-1}.$$

Clearly

$$\operatorname{Res}\left(\frac{-2i}{z^2+2az+1}, z_1\right) = \frac{-i}{\sqrt{a^2-1}}, \qquad \operatorname{Res}\left(\frac{-2i}{z^2+2az+1}, z_2\right) = \frac{i}{\sqrt{a^2-1}}.$$

Now we consider which roots lie in the unit disk $|z| < 1$. If $a > 1$, z_2 is not in the unit disk and z_1 lie in the unit disk. In this case, by residue theorem,

$$\int_{|z|=1}\frac{-2i}{z^2+2az+1}\,\mathrm{d}z = 2\pi i\frac{-i}{\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Similarly we can find the result when $a < -1$. We leave the arguments to readers.

Type II. Evaluate

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x.$$

Strategy: Fixing a $R > 0$ large enough, we construct a contour Γ_R by the following way. Starting from $-R$, we go along the real axis to the point R and then go from R back to $-R$ along the upper circle with radius R. We try to evalute

$$\int_{\Gamma_R} R(z)\,\mathrm{d}z$$

by residue theorem. On the other hand, we have

$$\int_{\Gamma_R} R(z)\,\mathrm{d}z = \int_{-R}^{R} R(x)\,\mathrm{d}x + \int_{|z|=R,\mathrm{im}(z)>0} R(z)\,\mathrm{d}z.$$

So if we take $R \to \infty$, the right-hand side above converges to

$$\int_{-\infty}^{\infty} R(x)\mathrm{d}x + \lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} R(z)\,\mathrm{d}z = 2\pi i \sum_{z_j} \mathrm{Res}(R(z), z_j),$$

where z_j is the singular points of $R(z)$ in the upper plane.

Example 2. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x.$$

solution: the function $1/(z^2+1)$ has one singularity on the upper half plane. That is i. The residue of $1/(z^2+1)$ at i equals to $-i/2$. Therefore it holds

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x + \lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} \frac{1}{z^2+1}\,\mathrm{d}z = \pi.$$

Now we evaluate the limit on the left-hand side above. By the parametrization $Re^{i\theta}$, we get

$$\lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} \frac{1}{z^2+1}\,\mathrm{d}z = \lim_{R\to\infty} \int_0^{\pi} \frac{1}{R^2e^{2i\theta}+1} Re^{i\theta} i\,\mathrm{d}\theta \leq \lim_{R\to\infty} \int_0^{\pi} \frac{R}{R^2-1}\mathrm{d}\theta = 0.$$

Therefore we know that

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \pi.$$

Type III. Evaluate

$$\mathrm{III.1} = \int_{-\infty}^{\infty} R(x)\cos x\,\mathrm{d}x, \qquad \mathrm{III.2} = \int_{-\infty}^{\infty} R(x)\sin x\,\mathrm{d}x.$$

Strategy: Evaluate

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,\mathrm{d}x. \tag{0.2}$$

Then III.1 and III.2 are real part and imaginary part of (0.2), respectively. To evaluate (0.2), we use the same contour Γ_R as in Type II. Therefore we know that

$$\int_{-\infty}^{\infty} R(x)e^{ix}\mathrm{d}x + \lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} R(z)e^{iz}\,\mathrm{d}z = 2\pi i \sum_{z_j} \mathrm{Res}\left(R(z)e^{iz}, z_j\right),$$

where z_j is the singular points of $R(z)e^{iz}$ in the upper plane.

Example 3. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,\mathrm{d}x.$$

Solution: Firstly $e^{iz}/(z^2+1)$ has one singularity i in the upper half plane. Moreover we can calculate

$$\mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, i\right) = \frac{e^{-1}}{2i}.$$

Clearly by the strategy, we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\mathrm{d}x + \lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} \frac{e^{iz}}{z^2+1}\,\mathrm{d}z = \frac{\pi}{e}.$$

Now we consider the limit on the left-hand side above. Using the parametrization $Re^{i\theta}$, we know that

$$\lim_{R\to\infty} \int_{|z|=R,\mathrm{Im}(z)>0} \frac{e^{iz}}{z^2+1}\,\mathrm{d}z = \lim_{R\to\infty} \int_0^{\pi} \frac{e^{-R\sin\theta}}{R^2e^{2i\theta}+1} e^{iR\cos\theta} Re^{i\theta} i\,\mathrm{d}\theta \leq \lim_{R\to\infty} \int_0^{\pi} \frac{R}{R^2-1}\,\mathrm{d}\theta = 0.$$

Here we used the fact that

$$e^{-R\sin\theta} \leq 1$$

since θ runs between 0 and π. Therefore the above arguments show that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,\mathrm{d}x = \frac{\pi}{e}.$$

Taking the real part of the left-hand side above, the problem is solved.

Type IV. Evaluate

$$\mathrm{IV.1} = \int_0^{\infty} R(x)\ln x\,\mathrm{d}x, \qquad \mathrm{IV.2} = \int_0^{\infty} x^{\alpha} R(x)\,\mathrm{d}x.$$

where $R(x)$ in IV.1 is an even function.

Strategy: These two integrals can be evaluated by the following contour. Firstly we choose the branch of $\ln z$ by eleminating the negative pure imaginary line. The argument runs from $-\pi/2$ to $3\pi/2$. Then we go from ϵ to R along the positive direction of the x-axis. Here ϵ is a small positive number and R is a large positive number. We proceed to go from R to $-R$ along the upper half circle $|z| = R$ and then go from $-R$ to $-\epsilon$ along the x-axis. Finally we jump over 0 by going from $-\epsilon$ to ϵ along the upper half circle $|z| = \epsilon$.

To evaluate IV.1 and IV.2, one just needs follow the arguments below.

Step 1. Letting $x = t^2$, IV.2 can be rewritten as

$$\mathrm{IV.2} = 2\int_0^{\infty} t^{2\alpha+1} R(t^2)\,\mathrm{d}t = 2\int_0^{\infty} e^{(2\alpha+1)\ln t} R(t^2)\,\mathrm{d}t.$$

Notice here we do this change of variable in order to make the $R(t^2)$ an even function with respect to t. If $R(x)$ in IV.2 is already an even number, this step can be skipped. Since we already assume $R(x)$ is an even function for IV.1, this step is not required for IV.1;

Step 2. By residue theorem we know that

$$\begin{aligned}\int_{\epsilon}^{R} R(x)\ln x\,\mathrm{d}x \quad &+ \quad \int_{|z|=R,\mathrm{Im}(z)>0} R(z)\ln z\,\mathrm{d}z + \int_{-R}^{-\epsilon} R(x)\ln x\,\mathrm{d}x \\ &+ \quad \int_{|z|=\epsilon,\mathrm{Im}(z)>0} R(z)\ln z\,\mathrm{d}z = 2\pi i\sum_j \mathrm{Res}\left(R(z)\ln z, z_j\right) \end{aligned} \tag{0.3}$$

where $\{z_j\}$ are singularities of $R(z)\ln z$ on the upper half plane. By the choice of the branch, we know that $\ln x = \ln|x| + i\pi$ for $x < 0$. Therefore the third integral on the left-hand side above can be rewritten as

$$\int_{-R}^{-\epsilon} R(x)\ln|x|\,\mathrm{d}x + i\pi\int_{-R}^{-\epsilon} R(x)\,\mathrm{d}x.$$

Applying change of variable, we get

$$\int_{-R}^{-\epsilon} R(x)\ln|x|\,\mathrm{d}x + i\pi\int_{-R}^{-\epsilon} R(x)\,\mathrm{d}x = \int_{\epsilon}^{R} R(y)\ln y\,\mathrm{d}y + i\pi\int_{\epsilon}^{R} R(y)\,\mathrm{d}y.$$

Applying the above equality to (0.3) and taking $\epsilon \to 0$, $R \to \infty$, we obtain

$$\begin{aligned}2\int_{0}^{\infty} R(x)\ln x\,\mathrm{d}x \quad &+ \quad \lim_{R\to\infty}\int_{|z|=R,\mathrm{Im}(z)>0} R(z)\ln z\,\mathrm{d}z + i\pi\int_{0}^{\infty} R(x)\,\mathrm{d}x \\ &+ \quad \lim_{\epsilon\to 0}\int_{|z|=\epsilon,\mathrm{Im}(z)>0} R(z)\ln z\,\mathrm{d}z = 2\pi i\sum_j \mathrm{Res}\left(R(z)\ln z, z_j\right)\end{aligned}$$

As for IV.2, after Step 1, we can apply residue theorem to get

$$\begin{aligned}\int_{\epsilon}^{R} e^{(2\alpha+1)\ln t}R(t^2)\,\mathrm{d}t \quad &+ \quad \int_{|z|=R,\mathrm{Im}(z)>0} e^{(2\alpha+1)\ln z}R(z^2)\,\mathrm{d}z + \int_{-R}^{-\epsilon} e^{(2\alpha+1)\ln t}R(t^2)\,\mathrm{d}t \\ &+ \quad \int_{|z|=\epsilon,\mathrm{Im}(z)>0} e^{(2\alpha+1)\ln t}R(t^2)\,\mathrm{d}z = 2\pi i\sum_j \mathrm{Res}\left(z^{2\alpha+1}R(z^2), c_j\right)\end{aligned} \tag{0.4}$$

where c_j are all singularities of $z^{2\alpha+1}R(z^2)$ in the upper half plane. Still by $\ln x = \ln|x| + i\pi$ $(x < 0)$, we can reduce the third integral above to

$$\int_{-R}^{-\epsilon} e^{(2\alpha+1)\ln|t|}e^{(2\alpha+1)i\pi}R(t^2)\,\mathrm{d}t = -e^{2\alpha\pi i}\int_{\epsilon}^{R} e^{(2\alpha+1)\ln s}R(s^2)\,\mathrm{d}s$$

Applying the above equality to (0.4) and taking $\epsilon \to 0$, $R \to \infty$, respectively, we get

$$\begin{aligned}\left(1 - e^{2\alpha\pi i}\right)\int_{\epsilon}^{R} e^{(2\alpha+1)\ln t}R(t^2)\,\mathrm{d}t \quad &+ \quad \lim_{R\to\infty}\int_{|z|=R,\mathrm{Im}(z)>0} e^{(2\alpha+1)\ln z}R(z^2)\,\mathrm{d}z \\ &+ \quad \lim_{\epsilon\to 0}\int_{|z|=\epsilon,\mathrm{Im}(z)>0} e^{(2\alpha+1)\ln t}R(t^2)\,\mathrm{d}z = 2\pi i\sum_j \mathrm{Res}\left(z^{2\alpha+1}R(z^2), c_j\right)\end{aligned} \tag{0.5}$$

Then the two integrals can be evaluated.

We now use one more example to complete the note

Example 4. Evaluate

$$\int_{0}^{\infty} \frac{x^{1/3}}{x^2+1}\,\mathrm{d}x.$$

Solution. Since $1/(x^2+1)$ is already an even function, we don't need Step 1 in the strategy. So by the contour, we know that

$$\int_{\epsilon}^{R} \frac{x^{1/3}}{x^2+1}\,\mathrm{d}x \quad + \quad \int_{|z|=R,\mathrm{Im}(z)>0} \frac{z^{1/3}}{z^2+1}\,\mathrm{d}z + \int_{-R}^{-\epsilon} \frac{x^{1/3}}{x^2+1}\,\mathrm{d}x \tag{0.6}$$

$$+ \quad \int_{|z|=\epsilon,\mathrm{Im}(z)>0} \frac{z^{1/3}}{z^2+1}\,\mathrm{d}z = 2\pi i \sum_j \mathrm{Res}\left(\frac{z^{1/3}}{z^2+1}, c_j\right).$$

1.

$$\left|\int_{|z|=R,\mathrm{Im}(z)>0} \frac{z^{1/3}}{z^2+1}\,\mathrm{d}z\right| \quad = \quad \left|\int_{|z|=R,\mathrm{Im}(z)>0} \frac{e^{1/3\ln z}}{z^2+1}\,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{e^{(\ln R)/3+(i\theta)/3}}{R^2e^{2i\theta}+1}\,Re^{i\theta} i\,\mathrm{d}\theta\right|$$

$$\leq \quad \left|\int_0^{\pi} \frac{R^{4/3}}{R^2-1}\,\mathrm{d}\theta\right| \longrightarrow 0, \qquad \text{as } R\to\infty.$$

2.

$$\left|\int_{|z|=\epsilon,\mathrm{Im}(z)>0} \frac{z^{1/3}}{z^2+1}\,\mathrm{d}z\right| \quad = \quad \left|\int_{|z|=\epsilon,\mathrm{Im}(z)>0} \frac{e^{1/3\ln z}}{z^2+1}\,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{e^{(\ln \epsilon)/3+(i\theta)/3}}{\epsilon^2e^{2i\theta}+1}\,\epsilon e^{i\theta} i\,\mathrm{d}\theta\right|$$

$$\leq \quad \left|\int_0^{\pi} \frac{\epsilon^{4/3}}{1-\epsilon^2}\,\mathrm{d}\theta\right| \longrightarrow 0, \qquad \text{as } \epsilon\to\infty.$$

3.

$$\int_{-R}^{\epsilon} \frac{x^{1/3}}{x^2+1} = \int_{-R}^{-\epsilon} \frac{e^{1/3\ln x}}{x^2+1} = \int_{-R}^{-\epsilon} \frac{e^{1/3(\ln|x|+i\pi)}}{x^2+1} = e^{i\pi/3}\int_{\epsilon}^{R} \frac{x^{1/3}}{x^2+1}.$$

Applying all the above arguments to (0.6), we get

$$(1+e^{i\pi/3})\int_0^{\infty} \frac{x^{1/3}}{x^2+1} = 2\pi i\,\mathrm{Res}\left(\frac{z^{1/3}}{z^2+1}, i\right) = 2\pi i \lim_{z\to i}\frac{z^{1/3}}{z+i} = 2\pi i\,\frac{i^{1/3}}{2i} = \pi e^{1/3\ln i} = \pi e^{i\pi/6}$$

therefore we know that

$$\int_0^{\infty} \frac{x^{1/3}}{x^2+1} = \frac{\pi}{e^{-i\pi/6}+e^{i\pi/6}} = \frac{\pi}{\sqrt{3}}.$$